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A note on divergence and flutter instabilities in elastic–plastic materials

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Abstract

Dynamic stability of uniform straining of a nonlinear elastic solid is known to require that all eigenvalues of the acoustic tensor associated with the tangent elastic moduli be real and nonnegative. The focus of this note is to what extent this conclusion applies to time-independent, elastoplastic materials. Nonlinearity of the elastic–plastic constitutive law imposes limits on validity of a solution to the linear problem for which the acoustic tensor is determined. The effect of those limits on the conclusions about instability is examined. © 2002 Elsevier Science Ltd. All rights reserved.

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1. Introduction

In a homogeneous material characterized by a *linear* constitutive relationship between the increments in stress and strain, the propagation of a disturbance represented by a planar wave depends critically on the eigenvalues of the acoustic tensor (Truesdell and Noll, 1965). The waves propagate with a real speed when all eigenvalues of the acoustic tensor for any wave front normal are real and positive. Occurrence of negative or complex eigenvalues is related to 'divergence' or 'flutter' growth of disturbances, respectively. The critical case when one eigenvalue is zero is associated with the onset of strain localization. These concepts are widely known; fundamental references are by Hadamard, Hill, Mandel and Rice (see, e.g. Rice, 1977).

In this note, we shall concentrate on the flutter instability in elastic–plastic materials with two or more constitutive branches (cones) of the incremental response; the divergence instability is also included in the analysis. Two sources of nonlinearity characterize plastic flow. One is related to the incremental nonlinearity of the tangent constitutive operator and the other to its dependence on the current state. The latter arises also in nonlinear elasticity, whereas the former is typical of the plastic behaviour. Despite the

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nonlinearities, the above classification with regard to the spectral properties of the acoustic tensor can be straightforwardly extended to the elastoplasticity by defining the acoustic tensor for the tangent stiffness moduli. Indeed, much effort has been directed towards determining the conditions for the onset of flutter (full references to this topic can be found in Bigoni and Loret (1999)). However, the interpretation of the instabilities in terms of the wave solution becomes less clear in view of the possibility of activating different constitutive cones at different material points, which would invalidate the wave solution itself. As a consequence, the physical interpretation of the occurrence of negative or complex eigenvalues of the acoustic tensor for an elastoplastic material becomes uncertain. A *reasonable conjecture* is that an arbitrary small disturbance in a uniformly strained material can grow and cross the domain of validity of the tangent stiffness moduli (e.g. by producing elastic unloading somewhere). But what happens *after* this is presently unknown. In fact, two opposite situations may be imagined, in which the perturbing oscillation may further amplify or, contrarily, damp. Only the former possibility would correspond to, say, a 'genuine' instability. Some evidence of flutter instability as triggered by the presence of a tractions-free boundary has been shown numerically by Simões (1997), but this single study is insufficient to draw definitive, general conclusions.

This note is aimed at providing certain rigorous statements which shed some more light on the meaning of flutter and divergence instabilities in incrementally nonlinear elastic–plastic solids. The two kinds of nonlinearity mentioned above are taken into account by introducing limits on validity of the solution to the linear problem for which the acoustic tensor is determined. The effect of these limits on the conclusions about instability is then examined. Illustrative examples are provided by combining the classical nonassociative plasticity with elastic anisotropy.

Notation: The notation employed essentially follows Gurtin (1981). In particular, the products of vectors or tensors are indicated as follows: $\mathbf{a}\cdot\mathbf{b} = a_k b_k$, $\mathbf{A}\cdot\mathbf{B} = A_{hk}B_{hk}$, $(\mathbf{Ab})_h = A_{hk}b_k$, $(\mathbf{AB})_{hk} = A_{ht}B_{tk}$, $(\mathbf{a}\otimes\mathbf{b})_{hk} = a_h b_k$, with the summation convention for repeated indices. The symbol i is reserved for the imaginary unit, $\mathrm{i} = \sqrt{-1}$. The Euclidean norm is denoted by $|\cdot|$, and $\mathbf{I}$ stands for the second-order identity tensor. The second-order tensor assigned by a fourth-order tensor C^{o} to a second-order tensor $\mathbf{A}$ is denoted by $\mathsf{C}^{\mathrm{o}}[\mathbf{A}]$: $(\mathsf{C}^{\mathrm{o}}[\mathbf{A}])_{st} = \mathsf{C}^{\mathrm{o}}_{sthk}A_{hk}$.

2. Problem formulation

Let us consider a homogeneous, arbitrarily anisotropic, plastically deforming material. The configuration of a material element at time $t = 0$ is taken as a fixed reference configuration for the Lagrangian description adopted below. In a given state, the incremental response of the material is characterized by the time-independent constitutive rate equation

$$\dot{\mathbf{S}} = \mathscr{C}(\dot{\mathbf{F}}), \tag{1}$$

where $\dot{\mathbf{S}}$ is the material time derivative of the first Piola–Kirchhoff stress tensor, $\dot{\mathbf{F}}$ is the material time derivative of the deformation gradient $\mathbf{F}$, or equivalently, $\dot{\mathbf{F}}$ is the velocity gradient in the reference configuration, and $\mathscr{C}$ is a state-dependent, nonlinear constitutive *operator*, positively homogeneous of degree one with respect to its argument. The conditions for objectivity of the constitutive law need not be discussed here.

Quantities that appear in the fundamental motion, whose stability is to be examined, are distinguished by a superscript o. We shall assume that the fundamental (Lagrangian) velocity gradient $\dot{\mathbf{F}}^{\mathrm{o}}$ corresponds to nonzero strain rate and lies inside a certain constitutive cone in $\dot{\mathbf{F}}$-space such that the operator $\mathscr{C}(\cdot)$ restricted to that cone becomes a linear operator, represented by a fourth-order, state-dependent tensor C^{o} of tangent moduli independent of $\dot{\mathbf{F}}$. For the purposes of this paper, it is convenient to use a weaker assumption specified as follows:

$$\mathscr{C}(\dot{\mathbf{F}}) = \mathsf{C}^{\mathrm{o}}[\dot{\mathbf{F}}] \quad \text{if } |\dot{\mathbf{F}} - \dot{\mathbf{F}}^{\mathrm{o}}| < m^{\mathrm{o}}|\dot{\mathbf{F}}^{\mathrm{o}}|, \tag{2}$$

where $m^{\mathrm{o}} \leqslant 1$ is a positive constant such that the inequality defines a neighborhood of $\dot{\mathbf{F}}^{\mathrm{o}}$ contained in the mentioned constitutive cone. Evidently, fulfillment of the inequality in Eq. (2) with m^{o} sufficiently small ensures that the strain-rate direction associated with $\dot{\mathbf{F}}$ is sufficiently close to the loading direction defined by $\dot{\mathbf{F}}^{\mathrm{o}}$.

The simplest illustration of Eq. (2) is provided by the constitutive law of the classical nonassociative elastoplasticity with smooth yield and plastic potential surfaces, where $\mathscr{C}$ at the yield point has two linear constitutive branches, singled out by the sign of the scalar product between $\dot{\mathbf{F}}$ and the normal, say $\mathbf{N}$, to the yield surface in $\mathbf{F}$-space, Fig. 1. More general constitutive laws for metal single crystals and polycrystals, with many yield surfaces intersecting at a vertex point, are also consistent with Eq. (2). Infinitely many yield surfaces may also correspond to a linear constitutive relationship in the cone of fully active loading (Hill, 1967).

Generally, the variations in C^{o} along a deformation path may be neglected only if the path length in the deformation-gradient space is sufficiently small, say, less than some positive constant l^{o}. Accordingly, we introduce the assumption that

$$\mathsf{C}^{\mathrm{o}} = \text{const} \quad \text{if } \int_0^t |\dot{\mathbf{F}}(\mathbf{x}, \tau)| \mathrm{d}\tau \leqslant l^{\mathrm{o}}. \tag{3}$$

We shall consider a plastically deforming, infinite, homogeneous medium, uniformly stressed at $t = 0$ in the assumed absence of body forces. The unboundedness of the domain, which allows boundary conditions to be left unspecified, is a useful working assumption in material stability considerations (Rice, 1977). The equations of motion are written in the basic and rate form as

$$\mathrm{Div}\,\mathbf{S} = \rho\ddot{\mathbf{u}}, \quad \mathrm{Div}\,\dot{\mathbf{S}} = \rho\dddot{\mathbf{u}}, \tag{4}$$

respectively. Here, ρ is the mass density and Div is the divergence operator, both taken in the reference configuration. The respective fundamental solution of the equations of motion (4), expressed in terms of displacements $\mathbf{u}$ from the position $\mathbf{x}$ of a material point in the *reference* configuration, is

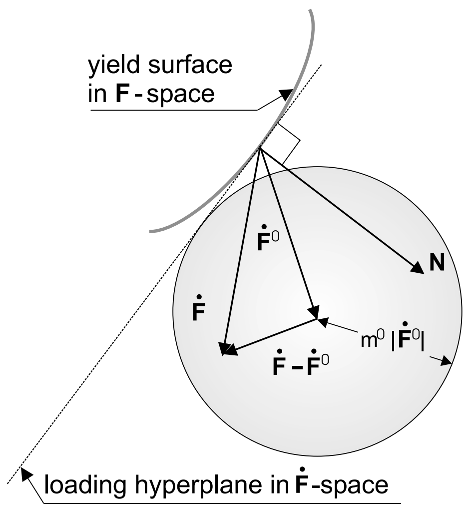

Fig. 1. Geometrical interpretation of the inequality in Eq. (2).

$$\mathbf{u}^{\mathrm{o}}(\mathbf{x},t) = t\dot{\mathbf{F}}^{\mathrm{o}}\mathbf{x}, \quad \dot{\mathbf{F}}^{\mathrm{o}} = \text{const}, \quad t \geqslant 0. \tag{5}$$

Physical validity of the fundamental solution is restricted by the requirement $\det \mathbf{F}^{\mathrm{o}} > 0$, with $\mathbf{F}^{\mathrm{o}} = \mathbf{I} + t\dot{\mathbf{F}}^{\mathrm{o}}$, which is ensured by Eq. (3) if l^{o} is sufficiently small.

As a version of the classical linear perturbation analysis, let us examine now a perturbed velocity field (defined in the reference configuration)

$$\mathbf{v}(\mathbf{x},t) = \mathbf{v}^{\mathrm{o}}(\mathbf{x}) + \gamma\mathbf{w}(\mathbf{x},t), \quad \mathbf{v}^{\mathrm{o}}(\mathbf{x}) = \dot{\mathbf{F}}^{\mathrm{o}}\mathbf{x}, \tag{6}$$

where $\mathbf{v} = \dot{\mathbf{u}}$, γ is a positive constant parameter and $\mathbf{w}$ is assumed in the standard form of a harmonic wave

$$\mathbf{w} = \operatorname{Re}\{\mathbf{a}\mathrm{e}^{\mathrm{i}k(\mathbf{n}\cdot\mathbf{x}\pm ct)}\}, \tag{7}$$

where $\mathrm{i} = \sqrt{-1}$, $\mathbf{n}$ is the unit vector of propagation in R^3, $\mathbf{a} \in C^3$ is the (nonzero) wave amplitude vector, $k \in R^+$ is the wave number, $c \in C$ is the wave speed. Here, R and C denote spaces of real and complex numbers, respectively, R^+ is the set of positive real numbers, and Re { } means 'real value of'. The constant γ is taken sufficiently small to ensure validity of the constitutive equation (2), at least in some initial time interval $[0, t^*]$.

Under the additional assumption (3) that $\mathsf{C}^{\mathrm{o}} = \text{const}$ in both space and time, the rate equation of motion (4) is satisfied by the velocity field (6) with Eq. (7) provided that

$$(\mathbf{A}^{\mathrm{o}}(\mathbf{n}) - \rho c^2\mathbf{I})\mathbf{a} = \mathbf{0}, \tag{8}$$

where $\mathbf{A}^{\mathrm{o}}(\mathbf{n})$ is the acoustic tensor related to the prescribed fundamental branch (2) of $\mathscr{C}$, defined by the identity $\mathbf{A}^{\mathrm{o}}(\mathbf{n})\mathbf{b} \equiv \mathsf{C}^{\mathrm{o}}[\mathbf{b}\otimes\mathbf{n}]\mathbf{n}$, to hold for every vector $\mathbf{b}$. Note that $\mathsf{C}^{\mathrm{o}}[\mathbf{a}\otimes\mathbf{n}]$ need not be equal to $\mathscr{C}(\mathbf{a}\otimes\mathbf{n})$ due to nonlinearity of the operator $\mathscr{C}$.

The eigenvalues ρc^2 and eigenvectors $\mathbf{a}$ of the acoustic tensor $\mathbf{A}^{\mathrm{o}}(\mathbf{n})$, which is nonsymmetric in general (as in the case of nonassociative elastoplasticity), may be real or complex conjugate. Let $c = \alpha + \mathrm{i}\beta$ and $\mathbf{a} = \mathbf{p} + \mathrm{i}\mathbf{q}$ satisfy Eq. (8), where α, β, $\mathbf{p}$, $\mathbf{q}$ are real. The eigenvector $\mathbf{a}$ is normalized so that its magnitude $|\mathbf{a}| = 1$, say. A perturbation (7) satisfying Eq. (4) is selected as a sum of two waves travelling in opposite directions

$$\mathbf{w}(\mathbf{x},t) = \mathrm{e}^{\beta kt}\{\mathbf{p}\cos[k(\mathbf{n}\cdot\mathbf{x}-\alpha t)] - \mathbf{q}\sin[k(\mathbf{n}\cdot\mathbf{x}-\alpha t)]\} + \mathrm{e}^{-\beta kt}\{\mathbf{p}\cos[k(\mathbf{n}\cdot\mathbf{x}+\alpha t)] - \mathbf{q}\sin[k(\mathbf{n}\cdot\mathbf{x}+\alpha t)]\}. \tag{9}$$

It is immediate to verify that for this choice the accelerations vanish at $t = 0$, i.e. $\dot{\mathbf{w}}(\mathbf{x},0) = \mathbf{0}$, which is consistent with $\operatorname{Div}\mathbf{S} = \mathbf{0}$ at $t = 0$. Note that this requirement might be overlooked if only the second equation in Eq. (4) were taken into account. The sign of β can be arbitrarily adjusted by replacing simply c by its conjugate. For a real wave speed, i.e. for $\beta = 0$, Eq. (9) describes a standing wave.

Suppose that the above perturbation in velocities is instantaneously superimposed at time $t = 0$ on the fundamental solution, with the initial condition $\mathbf{u}(\mathbf{x},0) = \mathbf{u}^{\mathrm{o}}(\mathbf{x},0)$ for displacements. Our aim is to examine the case when c^2 is not a nonnegative real number, i.e. when $\beta \neq 0$. If $\beta = 0$ then either $c^2 > 0$ which is the regular case of propagation of a harmonic wave, or $c = 0$ which corresponds to $\mathbf{w}$ independent of time in a dynamic solution and to bifurcation within a band of a quasi-static solution (Rice, 1977). Straightforward time integration of Eq. (9) yields the displacements in the subsequent motion free of further disturbances, in the form

$$\begin{aligned}\mathbf{u} = \mathbf{u}^{\mathrm{o}} + \frac{\gamma}{(\beta^2+\alpha^2)k}\big(&\mathrm{e}^{\beta kt}(\mathbf{r}\cos[k(\mathbf{n}\cdot\mathbf{x}-\alpha t)] - \mathbf{s}\sin[k(\mathbf{n}\cdot\mathbf{x}-\alpha t)]) - \mathrm{e}^{-\beta kt}(\mathbf{r}\cos[k(\mathbf{n}\cdot\mathbf{x}+\alpha t)] \\ &- \mathbf{s}\sin[k(\mathbf{n}\cdot\mathbf{x}+\alpha t)])\big), \quad \mathbf{r} = \beta\mathbf{p} - \alpha\mathbf{q}, \quad \mathbf{s} = \alpha\mathbf{p} + \beta\mathbf{q}, \quad \beta \neq 0.\end{aligned} \tag{10}$$

It should be noted that the validity of the above perturbed solution is limited by the requirement $\det \mathbf{F} > 0$ and by the assumptions (2) and (3) allowing us to perform the integration leading to Eq. (10) at fixed values of α, β, $\mathbf{p}$ and $\mathbf{q}$. Clearly, the condition $\beta \neq 0$ is necessary and sufficient for the existence of perturbations that amplify exponentially in time for every nonzero value of k. There are two cases when $\beta \neq 0$:

(A) some eigenvalue of $\mathbf{A}^{\mathrm{o}}(\mathbf{n})$ is real and negative; then $\alpha = 0$ (and $\mathbf{q} = \mathbf{0}$) which corresponds to monotonic growth of stationary waves ('divergence' instability), or
(B) two eigenvalues of $\mathbf{A}^{\mathrm{o}}(\mathbf{n})$ are complex conjugate; then $\alpha \neq 0$ (and $\mathbf{a}$ is complex) which corresponds to growth of oscillations in both space and time ('flutter' instability).

In contrast to a problem which is fully linear from the outset, this does not conclude the demonstration of instability for the material constitutive law (1) with the *state-dependent*, *nonlinear* constitutive operator $\mathscr{C}$. The following two important points remain to be analyzed.

- First, it remains to be shown that infinitesimal disturbances lead to finite deviations from the fundamental path without violating the condition of the constitutive linearization (2).
- Second, the finite deviations should be attainable within a sufficiently small increment in $\mathbf{F}^{\mathrm{o}}$ to satisfy the condition (3) of a fixed tangent moduli tensor C^{o}.

These two questions are examined in the next section.

3. Instability of uniform flow

3.1. Deformation paths of unrestricted length

In the stability analysis that follows, the velocity perturbation (9) superimposed on the fundamental motion will be considered for $\beta > 0$, since the sign of $\beta \neq 0$ is inessential. In view of the assumed sinusoidal form of all spatial perturbations in velocities, the amplitude factor $\gamma e^{\beta k t}$ plays the central role. When $\beta > 0$, a norm of the time-dependent spatial field $\mathbf{w}(\mathbf{x}, t)$ grows exponentially in time in the perturbed motion. This implies instability of the fundamental uniform flow (5) in the Lyapunov sense with respect to the velocity norm if the equations of motion (4) are treated as fully linear. In the case examined in this paper, the inequality in Eq. (2), related to the piecewise linearity of the constitutive rate equations, imposes a nonlinear constraint on the velocity gradient. Our first aim is to show that the conclusion drawn for $\beta \neq 0$ about the Lyapunov instability of the uniform flow (5) remains valid if the inequality constraint (2) is imposed while the time domain is left unbounded assuming $l^{\mathrm{o}} = \infty$ in Eq. (3). Mathematically, this is an expected result since the considerations of stability in the sense of Lyapunov may be a priori limited to a neighborhood of the fundamental motion. However, the implications in the context of incrementally nonlinear plasticity are less obvious, and therefore a detailed proof is provided below.

In the formal proof, it will be convenient to use the following seminorm of a vector field $\mathbf{b}(\mathbf{x})$ which is related to the constraint (2), namely

$$\|\mathbf{b}\| = \sup_{\mathbf{x}} |\nabla \mathbf{b}|, \tag{11}$$

where ∇ is the gradient operator in the reference configuration. On introducing the equivalence between any two sinusoidal fields that differ merely by a rigid translation in space, Eq. (11) provides a norm of $\mathbf{w}$. With that identification and for $\mathbf{n}$ and k fixed in time, any other spatial norm of a time-dependent sinusoidal spatial field (9) or (10) is equivalent to Eq. (11). Recall that two norms $\|\mathbf{w}\|$ and $\|\mathbf{w}\|'$ in the linear space of vector fields $\mathbf{w}$ are equivalent if and only if

$$m_1\|\mathbf{w}\| \leqslant \|\mathbf{w}\|' \leqslant m_2\|\mathbf{w}\|, \tag{12}$$

for some positive constants m_1, m_2. In those circumstances, we may restrict ourselves to examine stability with respect to the norm (11). Moreover, we will use the triangle inequality and its consequence

$$|\|\mathbf{w}_1\| - \|\mathbf{w}_2\|| \leqslant \|\mathbf{w}_1 + \mathbf{w}_2\| \leqslant \|\mathbf{w}_1\| + \|\mathbf{w}_2\|, \tag{13}$$

holding for every fields $\mathbf{w}_1$ and $\mathbf{w}_2$. On introducing the notation

$$M = M(\mathbf{a}, \mathbf{n}) = \sup_{\theta \in R} |\mathbf{p} \otimes \mathbf{n} \sin\theta + \mathbf{q} \otimes \mathbf{n} \cos\theta|, \tag{14}$$

and noticing that

$$M = \sup_{\mathbf{x}} |\mathbf{p} \otimes \mathbf{n} \sin[k(\mathbf{n} \cdot \mathbf{x} \pm \alpha t)] + \mathbf{q} \otimes \mathbf{n} \cos[k(\mathbf{n} \cdot \mathbf{x} \pm \alpha t)]|, \tag{15}$$

the seminorm (11) of the field $\mathbf{w}$ defined by Eq. (9) is bounded through Eq. (13) by

$$2Mk \sinh(\beta kt) \leqslant \|\mathbf{w}\|_t \leqslant 2Mk \cosh(\beta kt). \tag{16}$$

As a consequence, the distance in the sense of Eq. (11) between the fundamental and perturbed velocity solutions, Eqs. (5) and (6), at time t is bounded by

$$2Mk\gamma \sinh(\beta kt) \leqslant \|\mathbf{v} - \mathbf{v}^{\mathrm{o}}\|_t \leqslant 2Mk\gamma \cosh(\beta kt), \tag{17}$$

with $Mk\gamma > 0$ constant in time. From Eq. (9) it follows that the upper or lower bound is attained at instants t such that $\sin^2(k\alpha t) = 0$ or 1, respectively. For divergence instability, i.e. $\alpha = 0$, the upper bound gives the exact value of the distance at every instant.

Regarding for simplicity all quantities as nondimensionalized, the following statement on the Lyapunov instability of the fundamental solution (5) with respect to the velocity-gradient distance is obtained.

Proposition 1. *If, for some* $\mathbf{n}$, *not all eigenvalues of* $\mathbf{A}^{\mathrm{o}}(\mathbf{n})$ *are nonnegative real numbers and* $l^{\mathrm{o}} = \infty$ *in Eq.* (3) *then*

$$\exists\ \epsilon > 0\ \ \forall\ \delta > 0\ \exists\ \gamma,\ \ t^* > 0 : \|\mathbf{v} - \mathbf{v}^{\mathrm{o}}\|_{t=0} < \delta \wedge \|\mathbf{v} - \mathbf{v}^{\mathrm{o}}\|_{t=t^*} \geqslant \epsilon \tag{18}$$

and the inequality constraint in Eq. (2) *is satisfied for* $t \leqslant t^*$.

In words, Eq. (18) reads: There exists a positive number ϵ such that for every positive number δ, however small, there is a velocity perturbation whose norm is arbitrarily small initially ($\|\mathbf{v} - \mathbf{v}^{\mathrm{o}}\|_{t=0} < \delta$) and grows in a free dynamic motion to a finite value $\|\mathbf{v} - \mathbf{v}^{\mathrm{o}}\|_{t=t^*} \geqslant \epsilon$ reached at a certain time $t^* > 0$. All this occurs without violating the constraint in Eq. (2), while t^* is not bounded from above.

Proof. Take a positive $\epsilon < m^{\mathrm{o}}|\dot{\mathbf{F}}^{\mathrm{o}}|/\sqrt{2}$ (cf. Eq. (2)) [1] and for an arbitrary positive δ take a positive $\delta^* < \min(\delta, \epsilon)$. If some eigenvalue of $\mathbf{A}^{\mathrm{o}}(\mathbf{n})$ is not a real and nonnegative number then there exists $\beta > 0$ that defines the imaginary part of c. For an arbitrary $k > 0$, let

$$t^* = \frac{1}{k\beta} \sinh^{-1}\left(\frac{\epsilon}{\delta^*}\right), \quad \gamma = \frac{\delta^*}{2Mk}. \tag{19}$$

By inspection and with the help of Eq. (17), for these values of γ and t^* and for $\mathbf{w}$ defined by Eq. (9), we have

$$\|\mathbf{v} - \mathbf{v}^{\mathrm{o}}\|_{t=0} < \delta \wedge \|\mathbf{v} - \mathbf{v}^{\mathrm{o}}\|_{t=t^*} \geqslant \epsilon. \tag{20}$$

Moreover, from the upper bound in Eq. (17) it follows that

[1] The factor $1/\sqrt{2}$ is related to the ratio of the bounds in Eq. (17) which is $\coth(\beta kt) < \sqrt{2}$ provided $\sinh(\beta kt) > 1$.

$$|\dot{\mathbf{F}} - \dot{\mathbf{F}}^{\mathrm{o}}|_{\mathbf{x},t} \leqslant \|\mathbf{v} - \mathbf{v}^{\mathrm{o}}\|_t \leqslant 2Mk\gamma\cosh(\beta k t^*) \quad \text{for } t \leqslant t^*. \tag{21}$$

Recalling now that for every x there is $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$, and taking into account Eq. (19), we obtain

$$2Mk\gamma\cosh(\beta k t^*) = \epsilon + \frac{\delta^*}{\epsilon/\delta^* + \sqrt{(\epsilon/\delta^*)^2 + 1}} < \epsilon\sqrt{2} < m^{\mathrm{o}}|\dot{\mathbf{F}}^{\mathrm{o}}|. \tag{22}$$

Condition (2) implies the fulfillment of the inequality in Eq. (2) up to time t^* as required. This completes the proof of Proposition 1. □

In the above statement, the norm (11) can be replaced by any equivalent norm, which results straightforwardly by applying the inequalities (12).

It is clear that the conclusion about instability at fixed k requires an infinite time interval since $t^* \to \infty$ as $\delta \to 0$. Under the assumption of $\dot{\mathbf{F}}^{\mathrm{o}}$ fixed, an unbounded time interval corresponds to an unbounded length of a deformation path, so that the proof of Proposition 1 required the assumption $l^{\mathrm{o}} = \infty$ in Eq. (3).

The question arises whether instability can be proven with respect to a distance in displacement-gradient rather than in velocity-gradient, which would correspond to a finite departure of the perturbed motion from the fundamental path. In a fully linear problem an affirmative answer would be immediate. However, in view of the assumed constraint (2) on the velocity gradient, the initial disturbance in velocities cannot be replaced by another defined only in terms of displacements. This prompts us to retain the initial perturbation in velocities and to ask whether such arbitrarily small perturbation can induce a finite distance in displacement-gradient from the fundamental solution.

Application of Eq. (13) to Eq. (10) provides the estimates of the distance between the fundamental and perturbed solutions in displacements, measured in the sense of Eq. (11), viz.

$$2K\gamma\sinh(\beta k t) \leqslant \|\mathbf{u} - \mathbf{u}^{\mathrm{o}}\|_t \leqslant 2K\gamma\cosh(\beta k t), \tag{23}$$

with $K\gamma > 0$ constant in time, where

$$K = K(c, \mathbf{a}, \mathbf{n}) = \frac{1}{\beta^2 + \alpha^2} \sup_{\theta \in R} |\mathbf{r} \otimes \mathbf{n} \sin\theta + \mathbf{s} \otimes \mathbf{n}\cos\theta|, \tag{24}$$

with $\mathbf{r}$ and $\mathbf{s}$ defined in Eq. (10).

From Eq. (10) it follows that the lower or upper bound in Eq. (23) is reached at instants t such that $\sin^2(k\alpha t) = 0$ or 1, respectively. In the case of divergence instability, i.e. $\alpha = 0$, the lower bound gives an exact value of the distance at every instant.

The lower estimate (23) applied at the instant t^* defined by Eq. (19) in the proof of Proposition 1 yields

$$\|\mathbf{u} - \mathbf{u}^{\mathrm{o}}\|_{t^*} \geqslant \epsilon_1 = \frac{K}{M}\frac{\epsilon}{k}. \tag{25}$$

From Eq. (25) and Proposition 1 we obtain the proof of the following statement.

Proposition 2. *If, for some* $\boldsymbol{n}$, *not all eigenvalues of* $\boldsymbol{A}^{\mathrm{o}}(\boldsymbol{n})$ *are nonnegative real numbers and* $l^{\mathrm{o}} = \infty$ *in Eq.* (3) *then*

$$\exists\ \epsilon_1 > 0\ \ \forall\ \delta > 0\ \exists\ \gamma,\ \ t^* > 0 :\ \|\mathbf{v} - \mathbf{v}^{\mathrm{o}}\|_{t=0} < \delta \wedge \|\mathbf{u} - \mathbf{u}^{\mathrm{o}}\|_{t=t^*} \geqslant \epsilon_1 \tag{26}$$

and the inequality constraint in Eq. (2) *is satisfied for* $t \leqslant t^*$.

This shows the Lyapunov-type instability (on an infinite time domain) of the uniform flow with respect to two distances: the velocity-gradient distance $\|\mathbf{v} - \mathbf{v}^{\mathrm{o}}\|_{t=0}$ is used to measure the strength of an initial disturbance, and the displacement-gradient distance $\|\mathbf{u}-\mathbf{u}^{\mathrm{o}}\|_t$, equal to zero at $t=0$, is used to define the current distance between the fundamental and perturbed motions.

As in Proposition 1, the norm (11) can be replaced above by any equivalent norm of the spatially periodic fields involved.

As mentioned earlier, the above results are not unexpected. However, a too facile analogy to a fully linear problem contains a pitfall which is avoided here by using the perturbation (9) imposed on the fundamental velocity solution.

3.2. Short deformation paths

Instability established for an infinite time interval may have no physical meaning if the fundamental solution cannot be extended indefinitely in time. This is indeed the case for the plastic flow with a constant velocity gradient $\dot{\mathbf{F}}^{\mathrm{o}} \neq \mathbf{0}$ if a physical limit is imposed on the strain magnitude. Moreover, the tangent moduli tensor C^{o} will vary along a deformation path for any realistic material model. The variations may be neglected only if the path length in the deformation-gradient space is less than some value, denoted by l^{o} in Eq. (3), dependent on the desired accuracy of the constitutive description. Along the fundamental path (5), the current path length is $|\Delta\mathbf{F}^{\mathrm{o}}| = t|\dot{\mathbf{F}}^{\mathrm{o}}|$. In any perturbed motion that satisfies the inequality constraint in Eq. (2) up to an instant t^*, the path length is bounded by

$$\int_0^t |\dot{\mathbf{F}}(\mathbf{x},\tau)|\mathrm{d}\tau \leqslant t|\dot{\mathbf{F}}^{\mathrm{o}}| + \int_0^t |\dot{\mathbf{F}}(\mathbf{x},\tau) - \dot{\mathbf{F}}^{\mathrm{o}}|\mathrm{d}\tau < (1+m^{\mathrm{o}})|\dot{\mathbf{F}}^{\mathrm{o}}|t^* \quad \text{for } t \leqslant t^*. \tag{27}$$

The inequality constraint in Eq. (3) will thus be satisfied in any time interval $[0, t^*]$ such that

$$t^* \leqslant t^{\mathrm{o}} = \frac{l^{\mathrm{o}}}{(1+m^{\mathrm{o}})|\dot{\mathbf{F}}^{\mathrm{o}}|}. \tag{28}$$

The statement (18) can be extended to arbitrarily small t^* by appropriately adjusting the wave number k which was so far arbitrary. For any positive l^{o} and for an associated positive value of t^* satisfying Eq. (28), the first part of Eq. (19) can now be used to determine k instead of t^* as before. With this as the only change in the proof, from Proposition 1 we obtain the following corollary concerning instability of short deformation paths.

Proposition 3. *If, for some* $\boldsymbol{n}$, *not all eigenvalues of* $\boldsymbol{A}^{\mathrm{o}}(\boldsymbol{n})$ *are nonnegative real numbers then*

$$\exists\, \epsilon > 0 \quad \forall\, \delta > 0 \,\forall\, t^* \leqslant t^{\mathrm{o}}\, \exists\, k, \gamma > 0 : \|\mathbf{v}-\mathbf{v}^{\mathrm{o}}\|_{t=0} < \delta \wedge \|\mathbf{v}-\mathbf{v}^{\mathrm{o}}\|_{t=t^*} \geqslant \epsilon \tag{29}$$

and the inequality constraint in Eq. (2) *is satisfied for* $t \leqslant t^*$.

This means that the final inequality in Eq. (29) can be reached for l^{o} arbitrarily small, for any given fundamental deformation-rate $|\dot{\mathbf{F}}^{\mathrm{o}}|$. It should be noted that $\delta \to 0$ implies $k \to \infty$ (the short-wavelength limit) if a finite velocity-gradient distance measured by Eq. (11) is to be reached at a finite t^*.

The statement (29) expresses nothing else than the lack of continuous dependence of the velocity solution on initial data, with respect to the norm (11). This in turn is connected with the well-known concept of ill-posedness of the corresponding linear problem.

It may be interesting to note that it would be a flaw to formulate the statement (29) without any restriction on the velocity norm, just by analogy to the linear problem. Contrary to Proposition 1, the choice of a velocity norm in Proposition 3, where k can no longer be fixed in advance, is not fully arbitrary. [2] This can be seen from the inequality

$$\gamma \cosh(\beta k t) < \frac{1}{k} \frac{m^{o}}{2M} |\dot{\mathbf{F}}^{o}| \quad \text{for } t \leqslant t^{*}, \tag{30}$$

obtained from Eq. (22). It implies that the amplitude of $(\mathbf{v} - \mathbf{v}^{o})$ itself must tend to zero as $k \to \infty$, although the amplitude of the *gradient* of $(\mathbf{v} - \mathbf{v}^{o})$ at a given t^{*} can remain finite in the limit. In particular, the difference in kinetic energy within any bounded spatial domain tends to zero in the limit, which raises some doubts whether Proposition 3 may be interpreted as a proof of 'genuine' instability of plastic flow.

The next problem to be discussed is whether a property similar to Eq. (29) can be proven with respect to a distance in displacement-gradient rather than in velocity-gradient, cf. Proposition 2. Perhaps unexpectedly, we will show that such extension is *not* possible under the assumptions introduced above.

Observe first that under the inequality constraint in Eq. (2) satisfied up to time t^{*}, it is not possible to reach a given finite distance in the *displacement gradient* between the fundamental and perturbed motions within an arbitrarily small increment in $\mathbf{F}^{o}$. This can be seen from the following simple estimate

$$\|\mathbf{u} - \mathbf{u}^{o}\|_{t} \leqslant m^{o} |\Delta \mathbf{F}^{o}|_{t^{*}} \quad \text{for } t \leqslant t^{*}, \tag{31}$$

obtained analogously to Eq. (27). The possibility of reaching a finite value of $\|\mathbf{u} - \mathbf{u}^{o}\|$ at arbitrarily small $|\Delta \mathbf{F}^{o}|$ would be related to instability of *equilibrium*, which is not investigated here. The instability of plastic flow and instability of equilibrium are fundamentally different concepts for materials obeying an incrementally nonlinear constitutive law, [3] even if the plastic flow is regarded as quasi-static (Petryk, 1992).

Moreover, it turns out that the proof of Proposition 2 (carried out for $l^{o} = \infty$ in (3)) cannot even be extended to any *bounded* increment in $\mathbf{F}^{o}$, that is to any *finite* time interval $[0, t^{*}]$ at fixed $\dot{\mathbf{F}}^{o}$; this specifies limitations of the linear stability analysis in application to processes of plastic deformation. The same is shown for a finite l^{o} if we take any positive t^{*} satisfying Eq. (28) to fulfill the inequality in Eq. (3). Namely, the following statement holds true.

Proposition 4. *Irrespective of the type of eigenvalues of the acoustic tensor, if the inequality constraint in Eq.* (2) *is satisfied for* $t \leqslant t^{*}$ *then*

$$\forall\, t^{*} \leqslant t^{o} \ \forall\, \epsilon_{1} > 0 \ \exists\, \delta > 0 \ \forall\, k,\ \gamma > 0 : \|\mathbf{v} - \mathbf{v}^{o}\|_{t=0} < \delta \Rightarrow \|\mathbf{u} - \mathbf{u}^{o}\|_{t \leqslant t^{*}} < \epsilon_{1}. \tag{32}$$

Proof. The assumption $t^{*} < t^{o}$ in Eq. (32) implies, through Eq. (28), that Eq. (3) holds true, which jointly with the assumption (2), justifies the use of perturbation (9). Now, if $\beta = 0$ then the implication in Eq. (32) is trivial since in that case (9) describes a standing wave. It suffices thus to consider the case $\beta > 0$ which corresponds to a perturbed motion (10).

Suppose that Eq. (32) does not hold, so that

$$\exists\, t^{*} \leqslant t^{o} \ \exists\, \epsilon_{1} > 0 \ \forall\, \delta > 0 \ \exists\, k,\ \gamma > 0 \ \exists\, t \leqslant t^{*} : 2Mk\gamma < \delta \wedge \|\mathbf{u} - \mathbf{u}^{o}\|_{t} \geqslant \epsilon_{1}, \tag{33}$$

while the inequality constraint in Eq. (2) is satisfied for $t \leqslant t^{*}$.

[2] However, the norm (11) in Proposition 3 can still be replaced by any norm $\|\cdot\|'$ of the *velocity gradient* field such that Eq. (12) is satisfied for all k.

[3] The situation is different in the special case of an incrementally linear material where the inequality constraint in Eq. (2) is absent. Then the perturbation (7) can be superimposed on the degenerate fundamental motion of zero velocities, i.e. on an equilibrium state. In this case, the above established instability concerns equilibrium as well.

If $\delta \to 0$ then $\|\mathbf{u}-\mathbf{u}^{\rm o}\|_{t\leqslant t^*}$ tends also to zero by the upper bound in Eq. (23), in contradiction to Eq. (33), unless $k\to\infty$ or $k\to 0$.

If $k\to\infty$ as $\delta\to 0$ then from the lower bound in Eq. (17) and the inequality in Eq. (2), we have

$$\gamma\sinh(\beta kt) \leqslant \frac{\|\mathbf{v}-\mathbf{v}^{\rm o}\|_t}{2Mk} < \frac{m^{\rm o}|\dot{\mathbf{F}}^{\rm o}|}{2Mk}, \quad \text{for } t\leqslant t^*. \tag{34}$$

Hence, at each positive $t\leqslant t^*$, the lower bound in Eq. (23) tends to zero as $k\to\infty$. Since the ratio of the bounds in Eq. (23) tends to 1 in the limit, the upper bound in Eq. (23) tends to 0 at each positive $t\leqslant t^*$ as $\delta\to 0$, which contradicts Eq. (33).

Finally, if $k\to 0$ as $\delta\to 0$ then $\|\mathbf{u}-\mathbf{u}^{\rm o}\|_{t\leqslant t^*}$ tends to the lower bound in Eq. (23), by the continuity argument. But the lower bound tends to zero if $k\to 0$ as $\delta\to 0$, which leads again to the contradiction with Eq. (33). This completes the proof of Proposition 4. □

Hence, a finite value of $\|\mathbf{u}-\mathbf{u}^{\rm o}\|$ cannot be reached within a finite increment in $\mathbf{F}^{\rm o}$ in the class of perturbed motions (10) for arbitrarily small initial perturbations in velocities if the inequality constraint in Eq. (2) is satisfied. The violation of the constraint in Eq. (2), implied by the instability of motion in the sense of Proposition 3, may be followed later [4] by activation of constitutive branches other than the fundamental constitutive cone. No attempt is made here to analyze the subsequent nonlinear behaviour of the material. In other words, Proposition 4 does not imply that the fundamental solution is stable for a fully nonlinear constitutive law, rather, it shows that the boundary of the constitutive domain defined by inequalities in Eqs. (2) and (3) must be crossed before a finite deviation from the fundamental strain path is reached following an arbitrarily small disturbance.

4. Examples

4.1. Nonassociative elastoplasticity

The simple example of the nonlinear constitutive rate equation (1) is provided by the constitutive relationship of classical nonassociative elastoplasticity. When expressed in terms of the Eulerian strain rate $\mathbf{D}$ and an objective symmetric stress flux $\overset{\circ}{\mathbf{K}}$, it takes the well-known form

$$\overset{\circ}{\mathbf{K}} = \begin{cases} \mathsf{E}[\mathbf{D}] - \frac{1}{H} < \mathbf{Q}\cdot\mathsf{E}[\mathbf{D}] > \mathsf{E}[\mathbf{P}] & \text{if } f(\mathbf{K},\mathscr{K}) = 0, \\ \mathsf{E}[\mathbf{D}] & \text{if } f(\mathbf{K},\mathscr{K}) < 0, \end{cases} \tag{35}$$

where, for every scalar γ, $\langle\gamma\rangle = \max\{\gamma, 0\}$ is the operator which makes Eq. (35) incrementally nonlinear; moreover, f is the yield function, $\mathscr{K}$ denotes a collection of internal variables, $\mathbf{P}$ and $\mathbf{Q}$ are the normals to the smooth plastic potential and yield surfaces, respectively, E is the fourth-order tensor of current elastic tangent moduli (possessing both the major and minor symmetries), and the plastic modulus $H>0$ is related to the hardening modulus h via

$$H = h + \mathbf{Q}\cdot\mathsf{E}[\mathbf{P}]. \tag{36}$$

In particular, we can identify $\overset{\circ}{\mathbf{K}}$ with the Oldroyd derivative of the symmetric Kirchhoff stress $\mathbf{K}=\mathbf{S}\mathbf{F}^{\rm T}$,

$$\overset{\circ}{\mathbf{K}} = \dot{\mathbf{K}} - \mathbf{L}\mathbf{K} - \mathbf{K}\mathbf{L}^{\rm T}, \tag{37}$$

where $\mathbf{L}=\dot{\mathbf{F}}\mathbf{F}^{-1}$ is the spatial velocity gradient and $(\;)^{\rm T}$ denotes a transpose.

[4] That is, when the boundary of the fundamental constitutive cone in the strain-rate space is crossed by a perturbed solution.

At stress points on the current yield surface $f = 0$, the constitutive equation (1) corresponding to Eq. (35) reads

$$\dot{\mathbf{S}}\mathbf{F}^{\mathrm{T}} = \mathsf{E}[\mathbf{D}] + \mathbf{LK} - \frac{1}{H}\langle \mathbf{D}\cdot \mathsf{E}[\mathbf{Q}]\rangle \mathsf{E}[\mathbf{P}]. \tag{38}$$

Under the loading assumption $\mathbf{D}^{\mathrm{o}}\cdot \mathsf{E}[\mathbf{Q}] > 0$ in the fundamental motion, the equation (38) admits the linearized form corresponding to Eq. (2), viz.

$$\dot{\mathbf{S}}\mathbf{F}^{\mathrm{T}} = \mathsf{E}[\mathbf{D}] + \mathbf{LK} - \frac{1}{H}(\mathbf{D}\cdot \mathsf{E}[\mathbf{Q}])\mathsf{E}[\mathbf{P}] \quad \text{if } |\mathbf{D}-\mathbf{D}^{\mathrm{o}}| < m_1^{\mathrm{o}}|\mathbf{D}^{\mathrm{o}}|, \tag{39}$$

where we can take $m_1^{\mathrm{o}} = \mathbf{D}^{\mathrm{o}}\cdot \mathsf{E}[\mathbf{Q}]/(|\mathsf{E}[\mathbf{Q}]|\,|\mathbf{D}^{\mathrm{o}}|) \leqslant 1$. The fulfillment of the inequality in Eq. (39) with this value of m_1^{o} ensures that $\mathbf{Q}\cdot \mathsf{E}[\mathbf{D}] > 0$, which follows from

$$-\mathbf{D}\cdot \mathsf{E}[\mathbf{Q}] = (\mathbf{D}^{\mathrm{o}} - \mathbf{D})\cdot \mathsf{E}[\mathbf{Q}] - \mathbf{D}^{\mathrm{o}}\cdot \mathsf{E}[\mathbf{Q}] < m_1^{\mathrm{o}}|\mathsf{E}[\mathbf{Q}]|\,|\mathbf{D}^{\mathrm{o}}| - \mathbf{D}^{\mathrm{o}}\cdot \mathsf{E}[\mathbf{Q}] = 0.$$

An alternative graphical proof that the inequality in Eq. (39) implies $\mathbf{Q}\cdot \mathsf{E}[\mathbf{D}] > 0$ is shown in Fig. 2 (with the strain space defined relative to the current configuration). In turn, the inequality in Eq. (39) is readily implied by the fulfillment of the inequality condition in Eq. (2) with m^{o} sufficiently small, provided both $|\mathbf{F}|$ and $|\mathbf{F}^{-1}|$ are bounded. Explicitly, we can define $m^{\mathrm{o}} = m_1^{\mathrm{o}}|\mathbf{D}^{\mathrm{o}}|/|\mathbf{L}^{\mathrm{o}}| \leqslant m_1^{\mathrm{o}}$ if $\mathbf{F} = \mathbf{I}$.

Consequently, if the inequality conditions in Eqs. (2) and (3) are satisfied with l^{o} sufficiently small then no unloading occurs and the linearized constitutive relationship $\dot{\mathbf{S}} = \mathsf{C}^{\mathrm{o}}[\dot{\mathbf{F}}]$ is valid at each instant, with an explicit expression

$$\mathsf{C}^{\mathrm{o}} = \mathsf{G} + \mathbf{I}\boxtimes\mathbf{F}^{-1}\mathbf{S} - \frac{1}{H}\mathsf{G}[\mathbf{PF}]\otimes\mathsf{G}[\mathbf{QF}], \quad \mathsf{G} = (\mathbf{I}\boxtimes\mathbf{F}^{-1})\mathsf{E}(\mathbf{I}\boxtimes\mathbf{F}^{-\mathrm{T}}), \tag{40}$$

deduced from Eq. (38), where $\mathbf{F}^{-\mathrm{T}} = (\mathbf{F}^{-1})^{\mathrm{T}}$ and the tensor product $\boxtimes$ is defined such that $(\mathbf{A}\boxtimes\mathbf{B})[\mathbf{C}] = \mathbf{ACB}^{\mathrm{T}}$ for every second-order tensors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$.

Moreover, since H, $\mathbf{Q}$, $\mathbf{P}$ (and obviously $\mathbf{F}$ itself) vary in general along the fundamental path, the simplifying assumption in Eq. (3) that C^{o} is constant in time can only be physically admissible along deformation paths of sufficiently short length l^{o}. This provides some evidence that the mathematical results of Section 3.1 are only formal, while those of Section 3.2 may have physical meaning.

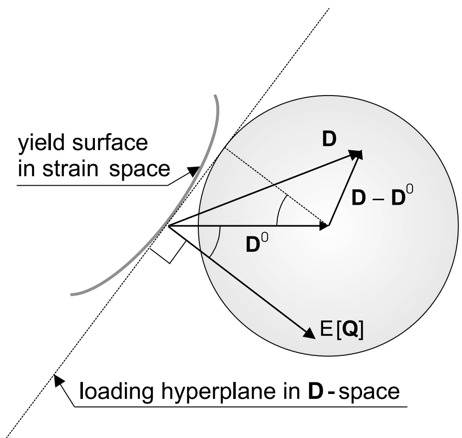

Fig. 2. Geometrical interpretation of the inequality in Eq. (39).

4.2. Anisotropic elasticity

Further specifications are adopted with the aim to discuss mainly the flutter instability, understood in the sense of Proposition 3 (however, with the reservations raised by Proposition 4). Following Bigoni and Loret (1999)—where the interested reader is referred to for details—we assume an anisotropic elastic law in the form

$$\mathsf{E} = \lambda \mathbf{B} \otimes \mathbf{B} + 2\mu \mathbf{B} \overline{\underline{\otimes}} \mathbf{B}, \tag{41}$$

where λ and μ are two material constants subject to the restrictions $\mu > 0$ and $3\lambda + 2\mu > 0$, $\mathbf{B}$ is a symmetric, positive defined second-order tensor and the tensor product $\overline{\underline{\otimes}}$ is defined in such a way that it assigns to three tensors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ the tensor $(\mathbf{A} \overline{\underline{\otimes}} \mathbf{B})[\mathbf{C}] = \mathbf{A}(\mathbf{C} + \mathbf{C}^{\mathrm{T}})\mathbf{B}^{\mathrm{T}}/2$. In addition, we assume a simple form for $\mathbf{B}$, namely

$$\mathbf{B} = b_1 \mathbf{b} \otimes \mathbf{b} + b_2(\mathbf{I} - \mathbf{b} \otimes \mathbf{b}), \tag{42}$$

describing a transversely isotropic elastic behaviour. In Eq. (42), the parameters b_1 and b_2 are assumed to depend on a single angular parameter $\hat{b}$ whose range is limited to $]0°, 90°[$ by the positive definiteness of $\mathbf{B}$:

$$b_1 = \sqrt{3}\cos\hat{b}, \quad b_2 = \sqrt{\frac{3}{2}}\sin\hat{b}. \tag{43}$$

It may be interesting to note that isotropic elasticity is recovered when $b_2 = b_1 = 1$, or $\hat{b} = \hat{b}_{\mathrm{iso}} \approx 54.74°$.

If the current configuration is taken as reference so that $\mathbf{F} = \mathbf{I}$ momentarily, the acoustic tensor corresponding to C^{o} in Eq. (40) is

$$\mathbf{A}^{\mathrm{ep}}(\mathbf{n}) = \mathbf{A}^{\mathrm{e}}(\mathbf{n}) - \frac{1}{H}\mathsf{E}[\mathbf{P}]\mathbf{n} \otimes \mathsf{E}[\mathbf{Q}]\mathbf{n}, \tag{44}$$

where $\mathbf{A}^{\mathrm{e}}(\mathbf{n})$ is the elastic acoustic tensor, defined as

$$\mathbf{A}^{\mathrm{e}}(\mathbf{n}) = (\lambda + \mu)\mathbf{Bn} \otimes \mathbf{Bn} + \mu(\mathbf{n} \cdot \mathbf{Bn})\mathbf{B} + (\mathbf{n} \cdot \mathbf{Kn})\mathbf{I}, \tag{45}$$

and

$$\mathsf{E}[\mathbf{H}]\mathbf{n} = \lambda\ (\mathbf{B} \cdot \mathbf{H})\mathbf{Bn} + 2\mu \mathbf{BHBn}, \quad \text{for } \mathbf{H} = \mathbf{P}, \mathbf{Q}. \tag{46}$$

Adopting Eqs. (44)–(46) we provide below necessary and sufficient conditions for flutter instability, identified with the existence of complex conjugate eigenvalues of $\mathbf{A}^{\mathrm{ep}}(\mathbf{n})$ for some $\mathbf{n}$, under the restrictive assumption that $\mathbf{B}$, $\mathbf{P}$ and $\mathbf{Q}$ share a common eigenvector, orthogonal to $\mathbf{b}$. With reference to Fig. 3, in particular, let us assume that $\mathbf{b}$ lies in the plane spanned by $\mathbf{k}_1$ and $\mathbf{k}_2$ that are two of unit eigenvectors $\mathbf{k}_i$, $i = 1, 2, 3$, of $\mathbf{K}$, so that $\mathbf{b}$ is singled out by the angle θ_σ. In addition, we look for propagation directions $\mathbf{n}$ lying on the plane spanned by $\mathbf{k}_1$ and $\mathbf{k}_2$ and, consequently, we denote by $\theta_{\mathbf{n}}$ the angle of inclination of $\mathbf{n}$ with respect to $\mathbf{k}_1$. As a consequence of the above assumptions, in the reference system $\mathbf{k}_i$ the acoustic tensor (44) has all elements in the third row and column null with the exception of

$$A_{33}^{\mathrm{ep}} = \mu(\mathbf{n} \cdot \mathbf{Bn})b_2 + \mathbf{n} \cdot \mathbf{Kn},$$

a quantity which we assume to be strictly positive. Taking the trace and the determinant of the remaining 2×2 matrix, we get the sum and the product of the in-plane eigenvalues a_i^{ep} of $\mathbf{A}^{\mathrm{ep}}(\mathbf{n})$

$$\begin{aligned} a_1^{\mathrm{ep}} + a_2^{\mathrm{ep}} &= a_1^{\mathrm{e}} + a_2^{\mathrm{e}} - \frac{1}{H}(f_1(\mathbf{n}) - f_2(\mathbf{n})), \\ a_1^{\mathrm{ep}} a_2^{\mathrm{ep}} &= a_1^{\mathrm{e}} a_2^{\mathrm{e}} + \frac{1}{H}\left(A_{nn}^{\mathrm{e}} f_2(\mathbf{n}) - A_{ss}^{\mathrm{e}} f_1(\mathbf{n}) + A_{ns}^{\mathrm{e}} f_3(\mathbf{n})\right), \end{aligned} \tag{47}$$

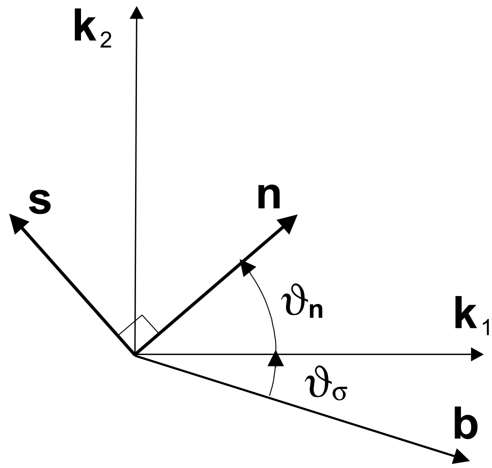

Fig. 3. Axis of elastic symmetry **b**, principal stress axes $\mathbf{k}_1$ and $\mathbf{k}_2$ and propagation direction **n**.

where a_1^e and a_2^e are the in-plane eigenvalues of the elastic acoustic tensor $\mathbf{A}^e(\mathbf{n})$ of components A^e_{nn}, A^e_{ss} and A^e_{ns} on the orthogonal unit vectors **n** and **s**, and

$$
\begin{aligned}
&f_1(\mathbf{n}) = (\mathbf{n}\cdot \mathsf{E}[\mathbf{P}]\mathbf{n})(\mathbf{n}\cdot \mathsf{E}[\mathbf{Q}]\mathbf{n}), \quad f_2(\mathbf{n}) = f_1(\mathbf{n}) - \mathsf{E}[\mathbf{P}]\mathbf{n}\cdot \mathsf{E}[\mathbf{Q}]\mathbf{n},\\
&f_3(\mathbf{n}) = (\mathbf{n}\cdot \mathsf{E}[\mathbf{P}]\mathbf{n})(\mathbf{s}\cdot \mathsf{E}[\mathbf{Q}]\mathbf{n}) + (\mathbf{s}\cdot \mathsf{E}[\mathbf{P}]\mathbf{n})(\mathbf{n}\cdot \mathsf{E}[\mathbf{Q}]\mathbf{n}).
\end{aligned}
\tag{48}
$$

From Eq. (47) and introducing the definitions:

$$
e(\mathbf{n}) = \frac{A^e_{ns}}{A^e_{nn} - A^e_{ss}}, \quad z(\mathbf{h}) = \frac{(\mathbf{h}\cdot\mathbf{n})^2 - (\mathbf{h}\cdot\mathbf{s})^2}{(\mathbf{h}\cdot\mathbf{n})(\mathbf{h}\cdot\mathbf{s})}, \qquad \text{for } \mathbf{h} = \mathsf{E}[\mathbf{P}]\mathbf{n}, \mathsf{E}[\mathbf{Q}]\mathbf{n},
\tag{49}
$$

we get the necessary and sufficient conditions for the existence of complex conjugate eigenvalues of $\mathbf{A}^{ep}(\mathbf{n})$ occurring at positive values of H:

$$
\begin{aligned}
&f_4(\mathbf{n}) \equiv 4 f_1 f_2 (1 - ez(\mathsf{E}[\mathbf{P}]\mathbf{n}))(1 - ez(\mathsf{E}[\mathbf{Q}]\mathbf{n})) > 0,\\
&\frac{A^e_{ns}}{e} f_5(\mathbf{n}) = \frac{A^e_{ns}}{e}(f_1 + f_2 + 2ef_3) > 0,\\
&H \in]H_-, H_+[, \quad H_\pm = \frac{A^e_{nn} - A^e_{ss}}{(a_1^e - a_2^e)^2}\left(f_5 \pm \sqrt{f_4}\right).
\end{aligned}
\tag{50}
$$

where [5] it may be noticed that $(a_1^e - a_2^e)^2 = (A^e_{nn} - A^e_{ss})^2 + (2A^e_{ns})^2$. Moreover, it can be observed from Eq. (50) that *the appearance of the complex eigenvalues of* $\mathbf{A}^{ep}(\mathbf{n})$ *is not influenced by the* '*geometrical term*' **LK** *in* Eq. (38) (for the adopted definition of $\overset{\circ}{\mathbf{K}}$), a conclusion obtained for isotropic elastic law by Bigoni and Zaccaria (1994) and limited here by the assumption that **n** belongs to the plane spanned by $\mathbf{k}_1$ and $\mathbf{k}_2$.

[5] Conditions (50) were obtained by Bigoni and Loret (1999), in a somewhat more restrictive context and employing a slightly different notation. Note that first part of Eqs. $(49)_2$ and (50) of Bigoni and Loret (1999) contain (inconsequential) misprints.

4.3. Drucker–Prager yield criterion

As a more specific example, we assume the Drucker–Prager yield criterion with deviatoric associativity, namely

$$\mathbf{P} = \cos\chi\hat{\mathbf{S}} + \frac{\sin\chi}{\sqrt{3}}\mathbf{I}, \qquad \mathbf{Q} = \cos\psi\hat{\mathbf{S}} + \frac{\sin\psi}{\sqrt{3}}\mathbf{I}, \tag{51}$$

where $\hat{\mathbf{S}} \in \text{Sym}$ is the unit norm deviatoric Cauchy stress.

The angular parameters ψ and χ describe the pressure-sensitivity and the dilatancy of the material, respectively. In addition to the above, we refer to uniaxial tension aligned with axis $\mathbf{k}_1$, so that the principal components of $\hat{\mathbf{S}}$ are $\{\sqrt{2/3}, -\sqrt{1/6}, -\sqrt{1/6}\}$. Assuming $\chi = 0$, $\psi = 30°$, $\hat{b} = 80°$, $\theta_\sigma = 15°$, $\lambda/\mu = 1$ and $H/\mu = 0.5$, and restricting the analysis to the infinitesimal theory, $\mathbf{LK} = \mathbf{0}$, Bigoni and Loret (1999) have shown that flutter occurs for a narrow interval of inclinations $\theta_\mathbf{n}$ ranging between $-8°$ and $-17°$. Now the same result holds true even when $\mathbf{LK}$ is not neglected in Eq. (38). However, *the real part* of eigenvalues of the acoustic tensor depend on $\mathbf{K}$. As an example, for $\theta_\mathbf{n} = -14°$, we obtain $a^{\text{ep}}_{1/2}/\mu = 0.2947 \pm 0.0533\text{i}$ and $a^{\text{ep}}_3/\mu = 0.3631$, for $\mathbf{K} = \mathbf{0}$; whereas we get: $a^{\text{ep}}_{1/2}/\mu = 0.7654 \pm 0.0533\text{i}$ and $a^{\text{ep}}_3/\mu = 0.8338$, for $\mathbf{K}$ having principal values equal to $\{0.5\mu, 0, 0\}$.

Now, even if H is assumed constant and the geometrical term is neglected, the acoustic tensor still depends on the current stress through $\mathbf{P}$ and $\mathbf{Q}$. Finally, we note that, when the geometrical term is neglected, the constitutive operator is positive definite for values of hardening modulus greater than $H^{\text{PD}}_{\text{cr}} = 0.4257\mu$. Therefore, in our case flutter may occur when the tangent constitutive operator is positive definite (so that strain localization is excluded within the 'small strain' theory).

4.4. Coulomb–Mohr yield criterion

We analyze here the special case of the Coulomb–Mohr yield criterion, when the stress state belongs to a plane of the yield surface at a finite distance from a corner. Adopting now small strain theory and $T_1 > T_3 > T_2$, where T_i, $i = 1, 2, 3$ are the principal components of Cauchy stress, the principal components of tensors $\mathbf{P}$ and $\mathbf{Q}$ (here nonnormalized) are

$$\{\mathbf{P}\} = \{1 + \sin\chi, -1 + \sin\chi, 0\}, \qquad \{\mathbf{Q}\} = \{1 + \sin\psi, -1 + \sin\psi, 0\}, \tag{52}$$

where χ and ψ are the dilatancy and friction angles, respectively. The present model does not satisfy deviatoric associativity, so that general results for flutter instability are not available. However, under the assumptions that the stress state is not in a vertex and that $\mathbf{n}$ lies in the plane 1–2, it is easy to shown that *flutter is excluded for isotropic elasticity*. On the other hand, flutter becomes possible for anisotropic elasticity of the type (41) and (42). In fact, taking $\chi = 0$, $\psi = 30°$, $\hat{b} = 80°$, $\theta_\sigma = 15°$, $\lambda/\mu = 1$ and $H/\mu = 2$ we find intervals of angle $\theta_\mathbf{n}$ where flutter occurs. As an example, for $\theta_\mathbf{n} = -14°$, we obtain $a^{\text{ep}}_{1/2}/\mu = 0.2962 \pm 0.0189\text{i}$ and $a^{\text{ep}}_3/\mu = 0.3631$.

We remark that even in this case, the tangent constitutive operator is positive definite, being $H^{\text{PD}}_{\text{cr}} = 1.7097\mu$.

The peculiarity of this example is that, assuming H constant, the acoustic tensor is constant in a finite neighborhood of the current state if the principal stress axes do not rotate with respect to the material (and if the geometric term is neglected). However, the fact that—in special situations—condition (3) can be verified for a finite value of l^o still does not mean that a 'genuine' instability with respect to a strain distance has been proved. Note in fact that Proposition (2) has been proved for infinite l^o, while for finite l^o and within the constitutive domain defined by inequalities in Eqs. (2) and (3) we have proved an opposite property expressed in Proposition (4).

The presented examples have been obtained exploiting the effects of anisotropic elasticity, therefore following an original idea of Bigoni and Loret (1999). They have shown in fact that anisotropic elasticity may make flutter instability possible even when this instability is excluded for elastic isotropy. Since elastic isotropy must be considered only a first approximation to the real behaviour of most materials, it may be expected that flutter instability conditions can be met more often than believed so far. Then, Proposition 3 indicates that numerical problems are likely to appear when simulating dynamic processes in nonassociative elastoplasticity. The constitutive models employed in the examples should be considered as simple prototypes, nevertheless they incorporate features characteristic of the behaviour of granular materials, as for instance pressure-sensitive yielding and plastic dilatancy. Also, an anisotropic elastic law of the type (41) was proposed as a measure of elastic induced anisotropy in sand (Gajo et al., 2001).

5. Discussion and conclusions

For a linear constitutive law, 'divergence' and 'flutter' instabilities (formally associated with the existence of negative or complex eigenvalues of the acoustic tensor) are related to monotonic or oscillatory growth of periodic initial disturbances in an infinite homogeneous medium (Rice, 1977). In the relevant literature it is often assumed that these concepts may be extended to elastoplastic materials (whose constitutive law is always incrementally nonlinear). The aim of the present note was to examine the mathematical justification of that extension.

For this purpose, spatially sinusoidal perturbations have been superimposed on the fundamental *velocity* field (of constant gradient), with the amplitude sufficiently small to activate only the fundamental constitutive branch. First, the constitutive operator has been assumed independent of the current state, but incrementally piecewise linear. In result, the (expected) Lyapunov instability of the fundamental motion defined on an *infinite* time interval (corresponding to deformation paths of unbounded length) has been proven for the incrementally nonlinear materials in which a linear relationship between the rates of stress and strain need hold merely in a certain neighborhood of the current velocity gradient (Propositions 1 and 2). Second, the assumption that the constitutive operator be independent of the current state has been relaxed and applied only to sufficiently short paths in the deformation gradient space. Then it has been shown that a vanishingly small initial perturbation of the fundamental velocity field grows to a finite perturbation of the velocity *gradient* (*not* of the velocity itself) in an arbitrarily short time interval if the superimposed wavelength is sufficiently short (Proposition 3). That rapid departure from the fundamental straining direction enables attaining a boundary of the domain of application of the fundamental tangent moduli within a small deformation increment. It is clear that the presence of an internal length scale in the material could influence that conclusion; cf. the analysis by Simões and Martins (1998) of flutter instability in friction problems.

However, the linear stability analysis on a finite time interval is inconclusive if the strain distance is adopted for deviations from the fundamental solution. Instead of instability which might be expected by analogy to the fully linear problem, in that case an opposite property has been proven irrespective of the type of eigenvalues of the acoustic tensor (Proposition 4). Further insight in the problem requires a study of solutions beyond the fundamental constitutive cone of the elastoplastic constitutive law, a task which falls beyond the scope of the present note.

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